Warm Up

Express each number in terms of $i$.

1. $\sqrt{-81}$  $9i$
2. $-\sqrt{-18}$  $-3i\sqrt{2}$

Find each complex conjugate.

3. $6 - i\sqrt{3}$  $6 + i\sqrt{3}$
4. $4i + \sqrt{2}$  $\sqrt{2} - 4i$

Find each product.

5. $(2 + \sqrt{2})(3 - \sqrt{2})$  $4 + \sqrt{2}$
6. $(6 + 3\sqrt{3})(1 + \sqrt{3})$  $15 + 9\sqrt{3}$
Objective

Perform operations with complex numbers.
Vocabulary

complex plane
absolute value of a complex number
Just as you can represent real numbers graphically as points on a number line, you can represent complex numbers in a special coordinate plane.

The **complex plane** is a set of coordinate axes in which the horizontal axis represents real numbers and the vertical axis represents imaginary numbers.
5-9 Operations with Complex Numbers

**Helpful Hint**

The real axis corresponds to the \( x \)-axis, and the imaginary axis corresponds to the \( y \)-axis. Think of \( a + bi \) as \( x + yi \).
Example 1: Graphing Complex Numbers

Graph each complex number.

A. $2 - 3i$

B. $-1 + 4i$

C. $4 + i$

D. $-i$
Check It Out! Example 1

Graph each complex number.

a. $3 + 0i$

b. $2i$

c. $-2 - i$

d. $3 + 2i$
Recall that absolute value of a real number is its distance from 0 on the real axis, which is also a number line. Similarly, the absolute value of an imaginary number is its distance from 0 along the imaginary axis.

**Absolute Value of a Complex Number**

<table>
<thead>
<tr>
<th>WORDS</th>
<th>ALGEBRA</th>
<th>EXAMPLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>The <strong>absolute value of a complex number</strong> ( a + bi ) is the distance from the origin to the point ((a, b)) in the complex plane, and is denoted (</td>
<td>a + bi</td>
<td>).</td>
</tr>
</tbody>
</table>
Example 2: Determining the Absolute Value of Complex Numbers

Find each absolute value.

A. \(|3 + 5i|\)
   \[\sqrt{3^2 + 5^2}\]
   \[\sqrt{9 + 25}\]
   \[\sqrt{34}\]

B. \(|-13|\)
   \[\sqrt{(-13)^2 + 0^2}\]
   \[\sqrt{169}\]
   \[13\]

C. \(|-7i|\)
   \[\sqrt{0^2 + (-7)^2}\]
   \[\sqrt{49}\]
   \[7\]
5-9 Operations with Complex Numbers

Check It Out! Example 2

Find each absolute value.

a. \( |1 - 2i| \)

\[
\sqrt{1^2 + (-2)^2} = \sqrt{1 + 4} = \sqrt{5}
\]

b. \( \left| \frac{1}{2} \right| \)

\[
\sqrt{\left( \frac{1}{2} \right)^2 + 0^2} = \sqrt{\frac{1}{4}} = \frac{1}{2}
\]

c. \( |23i| \)

\[
\sqrt{0^2 + 23^2} = \sqrt{529} = 23
\]
Adding and subtracting complex numbers is similar to adding and subtracting variable expressions with like terms. Simply combine the real parts, and combine the imaginary parts.

The set of complex numbers has all the properties of the set of real numbers. So you can use the Commutative, Associative, and Distributive Properties to simplify complex number expressions.
Helpful Hint

Complex numbers also have additive inverses. The additive inverse of \( a + bi \) is \(-(a + bi)\), or \(-a - bi\).
Example 3A: Adding and Subtracting Complex Numbers

Add or subtract. Write the result in the form $a + bi$.

\[(4 + 2i) + (-6 - 7i)\]

\[(4 - 6) + (2i - 7i)\]

\[-2 - 5i\]
Example 3B: Adding and Subtracting Complex Numbers

Add or subtract. Write the result in the form \(a + bi\).

\[(5 - 2i) - (-2 - 3i)\]

Distribute.

\[(5 - 2i) + 2 + 3i\]  \(\text{Add real parts and imaginary parts.}\)

\[(5 + 2) + (-2i + 3i)\]

\[7 + i\]
Example 3C: Adding and Subtracting Complex Numbers

Add or subtract. Write the result in the form $a + bi$.

$$(1 - 3i) + (-1 + 3i)$$

$$(1 - 1) + (-3i + 3i)$$

Add real parts and imaginary parts.

0
Check It Out! Example 3a

Add or subtract. Write the result in the form $a + bi$.

$(-3 + 5i) + (-6i)$

$(-3) + (5i - 6i)$

$-3 - i$
Check It Out! Example 3b

Add or subtract. Write the result in the form $a + bi$.

$2i - (3 + 5i)$

$(2i) - 3 - 5i$  
Add real parts and imaginary parts.

$(-3) + (2i - 5i)$  
Distribute.

$-3 - 3i$
Add or subtract. Write the result in the form \( a + bi \).

\[
(4 + 3i) + (4 - 3i)
\]

\[
(4 + 4) + (3i - 3i)
\]

Add real parts and imaginary parts.

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You can also add complex numbers by using coordinate geometry.
Example 4: Adding Complex Numbers on the Complex Plane

Find \((3 - i) + (2 + 3i)\) by graphing.

**Step 1** Graph \(3 - i\) and \(2 + 3i\) on the complex plane. Connect each of these numbers to the origin with a line segment.
Example 4 Continued

Find \((3 - i) + (2 + 3i)\) by graphing.

**Step 2** Draw a parallelogram that has these two line segments as sides. The vertex that is opposite the origin represents the sum of the two complex numbers, \(5 + 2i\). Therefore, \((3 - i) + (2 + 3i) = 5 + 2i\).
Example 4 Continued

Find \((3 - i) + (2 + 3i)\) by graphing.

**Check** Add by combining the real parts and combining the imaginary parts.

\[
(3 - i) + (2 + 3i) = (3 + 2) + (-i + 3i) = 5 + 2i
\]
Check It Out! Example 4a

Find \((3 + 4i) + (1 - 3i)\) by graphing.

**Step 1** Graph \(3 + 4i\) and \(1 - 3i\) on the complex plane. Connect each of these numbers to the origin with a line segment.
Step 2 Draw a parallelogram that has these two line segments as sides. The vertex that is opposite the origin represents the sum of the two complex numbers, $4 + i$. Therefore, $(3 + 4i) + (1 - 3i) = 4 + i$. 

Find $(3 + 4i) + (1 - 3i)$ by graphing.
Check It Out! Example 4a Continued

Find \((3 + 4i) + (1 - 3i)\) by graphing.

\textbf{Check} Add by combining the real parts and combining the imaginary parts.

\[(3 + 4i) + (1 - 3i) = (3 + 1) + (4i - 3i) = 4 + i\]
Find \((-4 - i) + (2 - 2i)\) by graphing.

**Step 1** Graph \(-4 - i\) and \(2 - 2i\) on the complex plane. Connect each of these numbers to the origin with a line segment.
Step 2 Draw a parallelogram that has these two line segments as sides. The vertex that is opposite represents the sum of the two complex numbers, \(-2 - 3i\). Therefore, 
\((-4 - i) + (2 - 2i) = -2 - 3i\).
Check It Out! Example 4b

Find \((-4 - i) + (2 - 2i)\) by graphing.

**Check** Add by combining the real parts and combining the imaginary parts.

\((-4 - i) + (2 - 2i) = (-4 + 2) + (-i - 2i) = -2 - 3i\)
You can multiply complex numbers by using the Distributive Property and treating the imaginary parts as like terms. Simplify by using the fact $i^2 = -1$. 
Example 5A: Multiplying Complex Numbers

Multiply. Write the result in the form $a + bi$.

$$-2i(2 - 4i)$$

$$-4i + 8i^2$$  Distribute.

$$-4i + 8(-1)$$  Use $i^2 = -1$.

$$-8 - 4i$$  Write in $a + bi$ form.
Example 5B: Multiplying Complex Numbers

Multiply. Write the result in the form $a + bi$.

$$(3 + 6i)(4 - i)$$

Multiply.

$12 + 24i - 3i - 6i^2$

Use $i^2 = -1$.

$12 + 21i - 6(-1)$

Write in $a + bi$ form.

$18 + 21i$
Example 5C: Multiplying Complex Numbers

Multiply. Write the result in the form $a + bi$.

$$ (2 + 9i)(2 - 9i) $$

$$ 4 - 18i + 18i - 81i^2 $$  \hspace{1cm} \text{Multiply.}

$$ 4 - 81(-1) $$  \hspace{1cm} \text{Use } i^2 = -1.

$$ 85 $$  \hspace{1cm} \text{Write in } a + bi \text{ form.}
Example 5D: Multiplying Complex Numbers

Multiply. Write the result in the form $a + bi$.

$(-5i)(6i)$

$-30i^2$ \hspace{1cm} Multiply.

$-30(-1)$ \hspace{1cm} Use $i^2 = -1$

$30$ \hspace{1cm} Write in $a + bi$ form.
Check It Out! Example 5a

Multiply. Write the result in the form \( a + bi \).

\[2i(3 - 5i)\]

\[6i - 10i^2\] Distribute.

\[6i - 10(-1)\] Use \( i^2 = -1 \).

\[10 + 6i\] Write in \( a + bi \) form.
Multiply. Write the result in the form $a + bi$.

$(4 - 4i)(6 - i)$

$24 - 4i - 24i + 4i^2$ \hspace{1cm} Distribute.

$24 - 28i + 4(-1)$ \hspace{1cm} Use $i^2 = -1$.

$20 - 28i$ \hspace{1cm} Write in $a + bi$ form.
Check It Out! Example 5c

Multiply. Write the result in the form \( a + bi \).

\((3 + 2i)(3 - 2i)\)

\[9 + 6i - 6i - 4i^2\]  \hspace{1cm} \text{Distribute.}

\[9 - 4(-1)\]  \hspace{1cm} \text{Use } i^2 = -1.

13  \hspace{1cm} \text{Write in } a + bi \text{ form.}
The imaginary unit $i$ can be raised to higher powers as shown below.

<table>
<thead>
<tr>
<th>Powers of $i$</th>
<th>$i^5 = i^4 \cdot i = 1 \cdot i = i$</th>
<th>$i^9 = i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$i^1 = i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$i^2 = -1$</td>
<td></td>
<td>$i^{10} = -1$</td>
</tr>
<tr>
<td>$i^3 = i^2 \cdot i = -1 \cdot i = -i$</td>
<td></td>
<td>$i^{11} = -i$</td>
</tr>
<tr>
<td>$i^4 = i^2 \cdot i^2 = -1 \cdot (-1) = 1$</td>
<td></td>
<td>$i^{12} = 1$</td>
</tr>
</tbody>
</table>

**Helpful Hint**

Notice the repeating pattern in each row of the table. The pattern allows you to express any power of $i$ as one of four possible values: $i$, $-1$, $-i$, or 1.
Example 6A: Evaluating Powers of $i$

Simplify $-6i^{14}$.

$-6i^{14} = -6(i^2)^7$ \hspace{1cm} Rewrite $i^{14}$ as a power of $i^2$.

$= -6(-1)^7$

$= -6(-1) = 6$ \hspace{1cm} Simplify.
Example 6B: Evaluating Powers of $i$

Simplify $i^{63}$.

$$i^{63} = i \cdot i^{62} \quad \text{Rewrite as a product of } i \text{ and an even power of } i.$$

$$= i \cdot (i^2)^{31} \quad \text{Rewrite } i^{62} \text{ as a power of } i^2.$$

$$= i \cdot (-1)^{31} = i \cdot -1 = -i \quad \text{Simplify.}$$
Check It Out! Example 6a

Simplify \( \frac{1}{2} i^7 \).

\[
\frac{1}{2} i^7 = \frac{1}{2} i \cdot i^6
\]

Rewrite as a product of \( i \) and an even power of \( i \).

\[
= \frac{1}{2} i \cdot (i^2)^3
\]

Rewrite \( i^6 \) as a power of \( i^2 \).

\[
= \frac{1}{2} i (-1)^3 = \frac{1}{2} \cdot -i = -\frac{1}{2} i
\]

Simplify.
Check It Out! Example 6b

Simplify \( i^{42} \).

\[
\begin{align*}
i^{42} &= (i^2)^{21} \\
&= (-1)^{21} \\
&= -1
\end{align*}
\]

Rewrite \( i^{42} \) as a power of \( i^2 \).
Simplify.
Recall that expressions in simplest form cannot have square roots in the denominator (Lesson 1-3). Because the imaginary unit represents a square root, you must rationalize any denominator that contains an imaginary unit. To do this, multiply the numerator and denominator by the complex conjugate of the denominator.

**Helpful Hint**

The complex conjugate of a complex number $a + bi$ is $a - bi$. (Lesson 5-5)
Example 7A: Dividing Complex Numbers

Simplify.

\[
\frac{3 + 10i}{5i}
\]\n
Multiply by the conjugate.

\[
\frac{3+10i}{5i} \cdot \left(\begin{array}{c}
-5i \\
-5i
\end{array}\right)
\]

\[
\frac{-15i - 50i^2}{-25i^2}
\]

Distribute.

\[
-15i + 50
\]

Use \(i^2 = -1\).

\[
\frac{-15i + 50}{25}
\]

\[
\frac{-3i + 10}{5} = 2 - \frac{3}{5}i
\]

Simplify.
Example 7B: Dividing Complex Numbers

Simplify.

\[
\frac{2 + 8i}{4 - 2i}
\]

Multiply by the conjugate.

\[
\frac{2 + 8i}{4 - 2i} \cdot \frac{4 + 2i}{4 + 2i}
\]

Distribute.

\[
\frac{8 + 4i + 32i + 16i^2}{16 - 4i^2}
\]

Use \(i^2 = -1\).

\[
\frac{-8 + 36i}{20}
\]

Simplify.

\[
\frac{-2 + 9i}{5} = -\frac{2}{5} + \frac{9}{5}i
\]
Check It Out! Example 7a

Simplify.

\[
\frac{3 + 8i}{-i} \cdot \frac{i}{i}
\]

Multiply by the conjugate.

\[
\frac{3i + 8i^2}{-i^2}
\]

Distribute.

\[
\frac{3i - 8}{1}
\]

Use \(i^2 = -1\).

Simplify.

\[-8 + 3i\]
Check It Out! Example 7b

Simplify.

\[
\frac{3 - i}{2 - i} \times \frac{2 + i}{2 + i}
\]

Multiply by the conjugate.

\[
\frac{6 - 2i + 3i - i^2}{4 - i^2}
\]

Distribute.

\[
\frac{7 + i}{5}
\]

Use \(i^2 = -1\).

\[
\frac{7}{5} + \frac{1}{5}i
\]

Simplify.
Lesson Quiz: Part I

Graph each complex number.

1. $-3 + 2i$

2. $4 - 2i$
Lesson Quiz: Part II

3. Find $|7 + 3i|$. $\sqrt{58}$

Perform the indicated operation. Write the result in the form $a + bi$.

4. $(2 + 4i) + (-6 - 4i) = -4$

5. $(5 - i) - (8 - 2i) = -3 + i$

6. $(2 + 5i)(3 - 2i) = 16 + 11i$

7. \( \frac{4 - 2i}{1 - i} \) = 3 + i

8. Simplify $i^{31}$. $-i$