6-6 Fundamental Theorem of Algebra

Warm Up

Lesson Presentation

Lesson Quiz
Warm Up
Identify all the real roots of each equation.

1. \(4x^5 - 8x^4 - 32x^3 = 0\) \(0, -2, 4\)
2. \(x^3 - x^2 + 9 = 9x\) \(1, -3, 3\)
3. \(x^4 + 16 = 17x^2\) \(-1, 1, -4, 4\)
4. \(3x^3 + 75x = 30x^2\) \(0, 5\)
Objectives

Use the Fundamental Theorem of Algebra and its corollary to write a polynomial equation of least degree with given roots.

Identify all of the roots of a polynomial equation.
You have learned several important properties about real roots of polynomial equations.

**The following statements are equivalent:**

<table>
<thead>
<tr>
<th>Statement</th>
<th>Description</th>
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<tbody>
<tr>
<td>A real number ( r ) is a root of the polynomial equation ( P(x) = 0 ).</td>
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<tr>
<td>( P(r) = 0 )</td>
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<tr>
<td>( r ) is an ( x )-intercept of the graph of ( P(x) ).</td>
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<tr>
<td>( x - r ) is a factor of ( P(x) ).</td>
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<tr>
<td>When you divide the rule for ( P(x) ) by ( x - r ), the remainder is 0.</td>
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<tr>
<td>( r ) is a zero of ( P(x) ).</td>
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You can use this information to write polynomial function when given in zeros.
Example 1: Writing Polynomial Functions

Write the simplest polynomial with roots \(-1, \frac{2}{3}, \) and 4.

\[ P(x) = (x + 1)(x - \frac{2}{3})(x - 4) \]

*If \( r \) is a zero of \( P(x) \), then \( x - r \) is a factor of \( P(x) \).*

\[ P(x) = (x^2 + \frac{1}{3}x - \frac{2}{3})(x - 4) \]

*Multiply the first two binomials.*

\[ P(x) = x^3 - \frac{11}{3}x^2 - 2x + \frac{8}{3} \]

*Multiply the trinomial by the binomial.*
Check It Out! Example 1a

Write the simplest polynomial function with the given zeros.

–2, 2, 4

\[ P(x) = (x + 2)(x - 2)(x - 4) \]

If \( r \) is a zero of \( P(x) \), then \( x - r \) is a factor of \( P(x) \).

\[ P(x) = (x^2 - 4)(x - 4) \]

Multiply the first two binomials.

\[ P(x) = x^3 - 4x^2 - 4x + 16 \]

Multiply the trinomial by the binomial.
Write the simplest polynomial function with the given zeros.

0, \( \frac{2}{3} \), 3

\[
P(x) = (x - 0)(x - \frac{2}{3})(x - 3)
\]

If \( r \) is a zero of \( P(x) \), then \( x - r \) is a factor of \( P(x) \).

\[
P(x) = (x^2 - \frac{2}{3}x)(x - 3)
\]

Multiply the first two binomials.

\[
P(x) = x^3 - \frac{11}{3}x^2 + 2x
\]

Multiply the trinomial by the binomial.
Notice that the degree of the function in Example 1 is the same as the number of zeros. This is true for all polynomial functions. However, all of the zeros are not necessarily real zeros. Polynomials functions, like quadratic functions, may have complex zeros that are not real numbers.

The Fundamental Theorem of Algebra

Every polynomial function of degree $n \geq 1$ has at least one zero, where a zero may be a complex number.

Corollary: Every polynomial function of degree $n \geq 1$ has exactly $n$ zeros, including multiplicities.
Using this theorem, you can write any polynomial function in factor form.

To find all roots of a polynomial equation, you can use a combination of the Rational Root Theorem, the Irrational Root Theorem, and methods for finding complex roots, such as the quadratic formula.
Example 2: Finding All Roots of a Polynomial

Solve \( x^4 - 3x^3 + 5x^2 - 27x - 36 = 0 \) by finding all roots.

The polynomial is of degree 4, so there are exactly four roots for the equation.

**Step 1** Use the rational Root Theorem to identify rational roots.

\[ \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36 \quad p = -36, \text{ and } q = 1. \]
**Example 2 Continued**

**Step 2** Graph \( y = x^4 - 3x^3 + 5x^2 - 27x - 36 \) to find the real roots.

*Find the real roots at or near \(-1\) and \(4\).*
Step 3  Test the possible real roots.

\[ \begin{array}{cccc}
-1 & 1 & -3 & 5 & -27 & -36 \\
 & -1 & 4 & -9 & 36 \\
1 & 1 & -4 & 9 & -36 & \text{0} \\
\end{array} \]

Test −1. The remainder is 0, so \((x + 1)\) is a factor.
Example 2 Continued

The polynomial factors into \((x + 1)(x^3 - 4x^2 + 9x - 36) = 0\).

\[
\begin{array}{c|cccc}
4 & 1 & -4 & 9 & -36 \\
\hline 
& 4 & 0 & 36 \\
& 1 & 0 & 9 & 0 \\
\end{array}
\]

Test 4 in the cubic polynomial. The remainder is 0, so \((x - 4)\) is a factor.
Example 2 Continued

The polynomial factors into \((x + 1)(x - 4)(x^2 + 9) = 0\).

**Step 4** Solve \(x^2 + 9 = 0\) to find the remaining roots.

\[
\begin{align*}
  x^2 + 9 &= 0 \\
  x^2 &= -9 \\
  x &= \pm 3i
\end{align*}
\]

The fully factored form of the equation is \((x + 1)(x - 4)(x + 3i)(x - 3i) = 0\). The solutions are 4, -1, 3i, -3i.
Solve \( x^4 + 4x^3 - x^2 + 16x - 20 = 0 \) by finding all roots.

The polynomial is of degree 4, so there are exactly four roots for the equation.

**Step 1** Use the rational Root Theorem to identify rational roots.

\[ \pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20 \]

\( p = -20, \) and \( q = 1. \)
Step 2  Graph \( y = x^4 + 4x^3 - x^2 + 16x - 20 \) to find the real roots.

Find the real roots at or near \(-5\) and \(1\).
Check It Out! Example 2 Continued

**Step 3** Test the possible real roots.

\[
\begin{array}{c|ccccc}
-5 & 1 & 4 & -1 & 16 & -20 \\
   &   & -5 & 5 & -20 & 20 \\
 1 & -1 & 4 & -4 & 0 \\
\end{array}
\]

*Test −5. The remainder is 0, so \((x + 5)\) is a factor.*
Check It Out! Example 2 Continued

The polynomial factors into \((x + 5)(x^3 - x^2 + 4x - 4) = 0\).

\[
\begin{array}{rrrr}
1 & 1 & -1 & 4 & -4 \\
\hline & 1 & 0 & 4 \\
\hline 1 & 0 & 4 & 0 \\
\end{array}
\]

Test 1 in the cubic polynomial. The remainder is 0, so \((x - 1)\) is a factor.
Check It Out! Example 2 Continued

The polynomial factors into \((x + 5)(x - 1)(x^2 + 4) = 0\).

**Step 4** Solve \(x^2 + 4 = 0\) to find the remaining roots.

\[
x^2 + 4 = 0 \\
x^2 = -2 \\
x = \pm 2i
\]

The fully factored form of the equation is \((x + 5)(x - 1)(x + 2i)(x - 2i) = 0\). The solutions are \(-5, 1, -2i, +2i\).
Example 3: Writing a Polynomial Function with Complex Zeros

Write the simplest function with zeros $2 + i, \sqrt{3} , \text{ and } 1$.

**Step 1** Identify all roots.

By the Rational Root Theorem and the Complex Conjugate Root Theorem, the irrational roots and complex come in conjugate pairs. There are five roots: $2 + i, 2 - i, \sqrt{3} , -\sqrt{3}, \text{ and } 1$. The polynomial must have degree 5.
Example 3 Continued

Step 2 Write the equation in factored form.

\[ P(x) = [x - (2 + i)][x - (2 - i)](x - \sqrt{3} )[(x - ( -\sqrt{3} ))](x - 1) \]

Step 3 Multiply.

\[
\begin{align*}
P(x) & = (x^2 - 4x + 5)(x^2 - 3)(x - 1) \\
& = (x^4 - 4x^3 + 2x^2 + 12x - 15)(x - 1) \\
P(x) & = x^5 - 5x^4 + 6x^3 + 10x^2 - 27x - 15
\end{align*}
\]
Write the simplest function with zeros $2i, 1 + \sqrt{2},$ and 3.

Step 1 Identify all roots.

By the Rational Root Theorem and the Complex Conjugate Root Theorem, the irrational roots and complex come in conjugate pairs. There are five roots: $2i, -2i, \ldots, \ldots,$ and 3. The polynomial must have degree 5.
**Check It Out! Example 3 Continued**

**Step 2** Write the equation in factored form.

\[ P(x) = (x - (2i))(x + (2i))(x - (1 + \sqrt{x}))(x - (1 - \sqrt{x}))(x - 3) \]

**Step 3** Multiply.

\[ P(x) = x^5 - 5x^4 + 9x^3 - 17x^2 + 20x + 12 \]
Example 4: Problem-Solving Application

A silo is in the shape of a cylinder with a cone-shaped top. The cylinder is 20 feet tall. The height of the cone is 1.5 times the radius. The volume of the silo is $828\pi$ cubic feet. Find the radius of the silo.

**Understand the Problem**

The cylinder and the cone have the same radius $x$. The answer will be the value of $x$.

**List the important information:**

- The cylinder is 20 feet tall.
- The height of the cone part is 1.5 times the radius, $1.5x$.
- The volume of the silo is $828\pi$ cubic feet.
Make a Plan

Write an equation to represent the volume of the body of the silo.

\[ V = V_{\text{cone}} + V_{\text{cylinder}} \]

\[ V(x) = \frac{1}{2} \pi x^3 + 20 \pi x^2 \]

Set the volume equal to \(828 \pi\).

\[ \frac{1}{2} \pi x^3 + 20 \pi x^2 = 828 \pi \]
Solve

$$\frac{1}{2} \, \pi x^3 + 20\pi x^2 - 828\pi = 0$$

$\frac{1}{2} \, x^3 + 20x^2 - 828 = 0$

The graph indicates a positive root of 6. Use synthetic division to verify that 6 is a root, and write the equation as $(x - 6)(\frac{1}{2}x^2 + 23x + 138) = 0$. The radius must be a positive number, so the radius of the silo is 6 feet.

Write in standard form.

Divide both sides by $\pi$.

$$\frac{1}{2} \, x^3 + 20x^2 - 828 = 0$$

\[
\begin{array}{c|cccc}
6 & \frac{1}{2} & 20 & 0 & -828 \\
\hline
& & 3 & 138 & 828 \\
\end{array}
\]

\[
\begin{array}{c|cccc}
\frac{1}{2} & 23 & 138 & 0 \\
\end{array}
\]
Look Back

Substitute 6 feet into the original equation for the volume of the silo.

\[ V(6) = \frac{1}{2} \pi (6)^3 + 20\pi (6)^2 \]

\[ V(6) = 828\pi \]
A grain silo is in the shape of a cylinder with a hemisphere top. The cylinder is 20 feet tall. The volume of the silo is $2106\pi$ cubic feet. Find the radius of the silo.

**Understand the Problem**

The cylinder and the hemisphere will have the same radius $x$. The answer will be the value of $x$.

**List the important information:**

- The cylinder is 20 feet tall.
- The height of the hemisphere is $x$.
- The volume of the silo is $2106\pi$ cubic feet.
Make a Plan

Write an equation to represent the volume of the body of the silo.

\[ V = V_{\text{hemisphere}} + V_{\text{cylinder}} \]

\[ V(x) = \frac{2}{3} \pi x^3 + 20\pi x^2 \]

Set the volume equal to \(2106\pi\).

\[ \frac{2}{3} \pi x^3 + 20\pi x^2 = 2106\pi \]
Solve

\[ \frac{2}{3} \pi x^3 + 20 \pi x^2 - 2106 \pi = 0 \]
\[ \frac{2}{3} x^3 + 20x^2 - 2106 = 0 \]

Write in standard form. Divide both sides by \( \pi \).

The graph indicates a positive root of 9. Use synthetic division to verify that 9 is a root, and write the equation as \((x - 9)(\frac{2}{3} x^2 + 26x + 234) = 0\). The radius must be a positive number, so the radius of the silo is 9 feet.
Look Back

Substitute 6 feet into the original equation for the volume of the silo.

\[ V(9) = \frac{2}{3} \pi(9)^3 + 20\pi(9)^2 \]

\[ V(9) = 2106\pi \] √
Lesson Quiz: Part I

Write the simplest polynomial function with the given zeros.

1. 2, –1, 1 \( x^3 - 2x^2 - x + 2 \)
2. 0, –2, \( x^4 + 2x^3 - 3x^2 - 6x \)
3. 2i, 1, –2 \( x^4 + x^3 + 2x^2 + 4x - 8 \)
4. Solve by finding all roots.
   \( x^4 - 5x^3 + 7x^2 - 5x + 6 = 0 \) 2, 3, i, –i
Lesson Quiz: Part II

5. The volume of a cylindrical vitamin pill with a hemispherical top and bottom can be modeled by the function \( V(x) = 10\pi r^2 + \frac{4}{3} \pi r^3 \), where \( r \) is the radius in millimeters. For what value of \( r \) does the vitamin have a volume of 160 mm\(^3\)? About 2 mm