Warm Up
Determine whether each system has zero, one or infinitely many solutions.

1. \[ \begin{align*}
3x + y &= 15 \\
3x - 2y &= 6
\end{align*} \] one

2. \[ \begin{align*}
x + 2y &= 18 \\
-2x - 4y &= -36
\end{align*} \] infinitely many

3. \[ \begin{align*}
2x + 3y &= 35 \\
4x + 6y &= 75
\end{align*} \] zero
Objectives

Find the determinants of $2 \times 2$ and $3 \times 3$ matrices.

Use Cramer’s rule to solve systems of linear equations.
Vocabulary

determinant
coefficient matrix
Cramer’s rule
Every square matrix \((n \text{ by } n)\) has an associated value called its determinant, shown by straight vertical brackets, such as \[
\begin{vmatrix}
1 & 2 \\
3 & 4
\end{vmatrix}
\]. The determinant is a useful measure, as you will see later in this lesson.

**Determinant of a 2 \(\times\) 2 Matrix**

<table>
<thead>
<tr>
<th>WORDS</th>
<th>NUMBERS</th>
<th>ALGEBRA</th>
</tr>
</thead>
</table>
| The **determinant** of a 2 by 2 matrix is the difference of the products of the diagonals. | \[
det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} =
\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (1)(4) - (3)(2) = -2
\] | \[
det\begin{bmatrix} a & b \\ c & d \end{bmatrix} =
\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb
\] |
The determinant of Matrix $A$ may be denoted as $\det A$ or $|A|$. Don’t confuse the $|A|$ notation with absolute value notation.
Example 1A: Finding the Determinant of a 2 x 2 Matrix

Find the determinant of each matrix.

\[
\begin{vmatrix}
1 & 5 \\
4 & 8 \\
\end{vmatrix}
\]

Find the difference of the cross products.

\[
\begin{vmatrix}
1 & 5 \\
4 & 8 \\
\end{vmatrix} = 1(8) - 4(5)
\]

\[
= 8 - 20
\]

The determinant is \(-12\).
Example 1B: Finding the Determinant of a 2 x 2 Matrix

Find the determinant of each matrix.

\[
\begin{vmatrix}
\frac{1}{2} & -1 \\
-\frac{1}{2} & 2
\end{vmatrix}
\]

\[
\frac{1}{2} \cdot (2) - \left( -\frac{1}{2} \right) \cdot (-1)
\]

\[
= 1 - \frac{1}{2}
\]

The determinant is \(\frac{1}{2}\).
Check It Out! Example 1a

Find the determinant of each matrix.

\[
\begin{vmatrix}
0.2 & 30 \\
-0.3 & 5 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
0.2 & 30 \\
-0.3 & 5 \\
\end{vmatrix}
= 0.2(5) - (-0.3)(30)
\]

\[
= 1 + 9
\]

The determinant is 10.
Check It Out! Example 1b

Find the determinant of each matrix.

\[
\begin{vmatrix}
1 & 3 \\
3 & 3 \\
5 & 3 \\
6 & 4 \\
\end{vmatrix}
\]

\[
= \frac{1}{3} \left( \frac{3}{4} \right) - \frac{5}{6} (3)
\]

\[
= \frac{3}{12} - \frac{15}{6} = \frac{3}{12} - \frac{30}{12} = -\frac{27}{12} = -2\frac{1}{4}
\]

The determinant is \(-2\frac{1}{4}\).
Check It Out! Example 1c

Find the determinant of each matrix.

\[
\begin{vmatrix}
1 & 1 \\ \\
2 & 8 \\ 4 & 2\pi
\end{vmatrix}
\]

\[
\begin{vmatrix}
1 & 1 \\ \\
2 & 8 \\ 4 & 2\pi
\end{vmatrix} = \frac{1}{2}(2\pi) - 4\left(\frac{1}{8}\right) = \pi - \frac{1}{2}
\]

The determinant is \(\pi - \frac{1}{2}\).
You can use the determinant of a matrix to help you solve a system of equations. For two equations with two variables written in \(ax + by = c\) form, you can construct a matrix of the coefficients of the variables.

For the system:

\[
\begin{align*}
\begin{bmatrix}
\begin{array}{cc}
a_1 & b_1 \\
a_2 & b_2
\end{array}
\end{bmatrix}
\end{align*}
\]

the coefficient matrix is

\[
\begin{bmatrix}
a_1 & b_1 \\
a_2 & b_2
\end{bmatrix}.
\]
The **coefficient matrix** for a system of linear equations in standard form is the matrix formed by the coefficients for the variables in the equations.

The determinant $D$ of the coefficient matrix is

$$
\begin{vmatrix}
  a_1 & b_1 \\
  a_2 & b_2
\end{vmatrix}.
$$

**Cramer’s Rule for Two Equations**

\[
\begin{align*}
  a_1x + b_1y &= c_1 \\
  a_2x + b_2y &= c_2
\end{align*}
\]

has solutions

\[
\begin{align*}
  x &= \frac{\begin{vmatrix}
    c_1 & b_1 \\
    c_2 & b_2
  \end{vmatrix}}{D}, \\
  y &= \frac{\begin{vmatrix}
    a_1 & c_1 \\
    a_2 & c_2
  \end{vmatrix}}{D}, \text{ where } D = \begin{vmatrix}
    a_1 & b_1 \\
    a_2 & b_2
  \end{vmatrix}.
\end{align*}
\]
You can use Cramer’s rule to tell whether the system represented by the matrix has one solution, no solution, or infinitely many solutions.

<table>
<thead>
<tr>
<th>Solutions of Systems</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $D \neq 0$, the system is consistent and has <strong>one</strong> unique solution.</td>
</tr>
<tr>
<td>If $D = 0$ and <strong>at least one</strong> numerator determinant is 0, the system is dependent and has <strong>infinitely many</strong> solutions.</td>
</tr>
<tr>
<td>If $D = 0$ and <strong>neither</strong> numerator determinant is 0, the system is inconsistent and has no solution.</td>
</tr>
</tbody>
</table>
Example 2A: Using Cramer’s Rule for Two Equations

Use Cramer’s rule to solve each system of equations.

\[
\begin{align*}
2x + y &= 10 \\
3x - 2y &= 8
\end{align*}
\]

**Step 1** Find \( D \), the determinant of the coefficient matrix.

\[
D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} = 2(-2) - 1(3) = -7
\]

\( D \neq 0 \), so the system is consistent.
Example 2A Continued

**Step 2** Solve for each variable by replacing the coefficients of that variable with the constants as shown below.

\[
\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = \begin{vmatrix} 10 & 1 \\ 8 & -2 \end{vmatrix} = 4
\]

\[
D = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = \begin{vmatrix} 2 & 10 \\ 3 & 8 \end{vmatrix} = 2
\]

The solution is \((4, 2)\).
Example 2B: Using Cramer’s Rule for Two Equations

Use Cramer’s rule to solve each system of equations.

\[
\begin{align*}
3x + 5 &= 2y \\
15 - 6y &= -9x
\end{align*}
\]

**Step 1** Write the equations in standard form.

\[
\begin{align*}
3x - 2y &= -5 \\
9x - 6y &= -15
\end{align*}
\]
Example 2B Continued

Step 2 Find the determinant of the coefficient matrix.

\[ D = \begin{vmatrix} 3 & -2 \\ 9 & -6 \end{vmatrix} = 3(-6) - 9(-2) = 0 \]

\[ D = 0, \text{ so the system is either inconsistent or dependent. Check the numerators for } x \text{ and } y \text{ to see if either is 0.} \]

\[ \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \rightarrow \begin{vmatrix} -5 & -2 \\ -15 & -6 \end{vmatrix} = 0 \]

\[ \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \rightarrow \begin{vmatrix} 3 & -5 \\ 9 & -15 \end{vmatrix} = 0 \]

Since at least one numerator is 0, the system is dependent and has infinitely many solutions.
Use Cramer’s rule to solve. \[
\begin{aligned}
6x - 2y &= 14 \\
3x &= y + 7
\end{aligned}
\]

Step 1 Write the equations in standard form.
\[
\begin{aligned}
6x - 2y &= 14 \\
3x - y &= 7
\end{aligned}
\]
Check It Out! Example 2 Continued

Step 2 Find the determinant of the coefficient matrix.

\[ D = \begin{vmatrix} 6 & -2 \\ 3 & -1 \end{vmatrix} = 6(-1) - 3(-2) = 0 \]

\( D = 0 \), so the system is either inconsistent or dependent. Check the numerators for \( x \) and \( y \) to see if either is 0.

\[ X = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \rightarrow \begin{vmatrix} 14 & -2 \\ 7 & -1 \end{vmatrix} = 0 \]

Because \( D = 0 \) and one of the numerator determinants is equal to 0, the system is dependent and has infinitely many solutions.
To apply Cramer’s rule to $3 \times 3$ systems, you need to find the determinant of a $3 \times 3$ matrix. One method is shown below.

Rewrite the first two columns at the right side of the determinant. Add the sum of the products of the red diagonals. Then subtract the sum of the blue diagonals.

\[
\begin{vmatrix}
  a_1 & b_1 & c_1 \\
  a_2 & b_2 & c_2 \\
  a_3 & b_3 & c_3 \\
\end{vmatrix} = a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)
\]
Example 3: Finding the Determinant of a $3 \times 3$ Matrix

Find the determinant of $M$.

$$M = \begin{bmatrix} 2 & 4 & 1 \\ 5 & 2 & 3 \\ 1 & 4 & 8 \end{bmatrix}$$

$$\text{det } M = \begin{vmatrix} 2 & 4 & 1 \\ 5 & 2 & 3 \\ 1 & 4 & 8 \end{vmatrix}, \text{ so write }$$

Step 1 Multiply each “down” diagonal and add.

$$2(2)(8) + 4(3)(1) + 1(5)(4) = 64$$

Step 2 Multiply each “up” diagonal and add.

$$1(2)(1) + 4(3)(2) + 8(5)(4) = 186$$
Example 3 Continued

Step 3 Find the difference of the sums.

\[64 - 186 = -122\]

The determinant is \(-122\).

**Check Use a calculator.**

\[
\begin{bmatrix}
2 & 4 & 1 \\
5 & 2 & 3 \\
1 & 4 & 8
\end{bmatrix}
\]

\[\text{det}([A]) = -122\]
Check It Out! Example 3

Find the determinant of

\[
\begin{vmatrix}
2 & -3 & 4 \\
5 & 1 & -2 \\
10 & 3 & -1
\end{vmatrix}
\]

**Step 1** Multiply each “down” diagonal and add.

\[
2(1)(-1) + (-3)(-2)(10) + 4(5)(3) = 118
\]

**Step 2** Multiply each “up” diagonal and add.

\[
10(1)(4) + (3)(-2)(2) - 1(5)(-3) = 43
\]
Check It Out! Example 3 Continued

**Step 3** Find the difference of the sums.

\[ 118 - 43 = 75 \]

The determinant is 75.

**Check** Use a calculator.

\[
\begin{bmatrix}
2 & -3 & 4 \\
5 & 1 & -2 \\
10 & 3 & -1
\end{bmatrix}
\]

\[
det([A]) = 75
\]
Cramer’s rule can be expanded to cover $3 \times 3$ systems.

Cramer’s Rule for Two Equations

\[
\begin{align*}
\begin{cases}
 a_1 x + b_1 y &= c_1 \\
 a_2 x + b_2 y &= c_2 
\end{cases}
\end{align*}
\]

has solutions

\[
x = \frac{c_1 b_2 - c_2 b_1}{D},
\]

\[
y = \frac{a_1 c_2 - a_2 c_1}{D},
\]

where

\[
D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}.
\]

If $D \neq 0$, then the system has a unique solution.

If $D = 0$ and no numerator is 0, then the system is inconsistent. If $D = 0$ and at least one numerator is 0, then the system may be inconsistent or dependent.
Example 4: Nutrition Application

A nutritionist creates a diet for a long-distance runner that includes 3400 Calories from 680 grams of food, with half the Calories coming from carbohydrates. How many grams of protein, carbohydrates, and fat will this diet include?

<table>
<thead>
<tr>
<th>Calories per Gram</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Food</td>
<td>Calories</td>
</tr>
<tr>
<td>Protein</td>
<td>4</td>
</tr>
<tr>
<td>Carbohydrates</td>
<td>4</td>
</tr>
<tr>
<td>Fat</td>
<td>9</td>
</tr>
</tbody>
</table>

The diet will include \( p \) grams of protein, \( c \) grams of carbohydrates, and \( f \) grams of fat.
Example 4 Continued

\[4p + 4c + 9f = 3400\]  \hspace{1cm} \text{Equation for total Calories}

\[p + c + f = 680\]  \hspace{1cm} \text{Total grams of food}

\[4c = 1700\]  \hspace{1cm} \text{Calories from carbohydrates,} \quad \frac{1}{2}(3400) = 1700

Use a calculator.  \hspace{1cm} D = \begin{vmatrix} 4 & 4 & 9 \\ 1 & 1 & 1 \\ 0 & 4 & 0 \end{vmatrix} = 20
The diet includes 119 grams of protein, 425 grams of carbohydrates, and 136 grams of fat.
Check It Out! Example 4

What if...? A diet requires 3200 calories, 700 grams of food, and 70% of the Calories from carbohydrates and fat. How many grams of protein, carbohydrates, and fat does the diet include?

<table>
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<tr>
<th>Calories per Gram</th>
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<td>Fat</td>
<td>9</td>
</tr>
</tbody>
</table>

The diet will include \( p \) grams of protein, \( c \) grams of carbohydrates, and \( f \) grams of fat.
Check It Out! Example 4 Continued

\[4p + 4c + 9f = 3200\]  \(\text{Equation for total Calories}\)

\[p + c + f = 700\]  \(\text{Total grams of food}\)

\[4c + 9f = 2240\]  \(\text{Calories from carbohydrates and fat,}\)
\[\text{70}\% (3200) = 2240.\]

Use a calculator.  \[D = \begin{vmatrix} 4 & 4 & 9 \\ 1 & 1 & 1 \\ 0 & 4 & 9 \end{vmatrix} = 20\]
The diet includes 240 grams of protein, 380 grams of carbohydrates, and 80 grams of fat.
Lesson Quiz

Find the determinant of each matrix.

1. \[
\begin{bmatrix}
  6 & 1.5 \\
  10 & 3.5 
\end{bmatrix}
\]
2. \[
\begin{bmatrix}
  3 & 1 & -4 \\
  2 & 0 & -1 \\
  1 & 5 & 4 
\end{bmatrix}
\]

Use Cramer’s rule to solve.

\[
\begin{align*}
4x + 3y &= 30 \\
5x - 6y &= 31
\end{align*}
\]

\[x = 7; y = \frac{2}{3}\]

4. Jeff buys 7 apples and 4 pears for $7.25. At the same prices, Hayley buy 5 apples and 9 pears for $10.40. What is the price of one pear?

$0.85